Approximation of the Unsteady Brinkman-Forchheimer Equations by the Pressure Stabilization Method

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In this work, we propose and analyze the pressure stabilization method for the unsteady incompressible Brinkman-Forchheimer equations. We present a time discretization scheme which can be used with any consistent finite element space approximation. Second-order error estimate is proven. Some numerical results are also given. Numer Methods Partial Differential Eq 33: 1949–1965, 2017

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I. INTRODUCTION

The flow of fluids through porous media at high Reynolds numbers is often encountered in chemical, petroleum and ground-water engineering, as well as in many other industrial applications. The place of momentum equations is occupied by the experimental observations summarized mathematically as Darcy's law. However, Darcy's law breaks down under conditions of high velocity flow. An extension to the traditional form of Darcy's law is the Brinkman term, which is used to account for transitional flow between boundaries. For very high velocities, inertial effects become significant. An inertial term is added to Darcy's equation, known as the Forchheimer term.

In this work, we are concerned with the following Darcy-Brinkman-Forchheimer (DBF) equations:

$$\begin{aligned} \boldsymbol{u}_t - \gamma \Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega_T, \\ \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega_T, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Sigma_T \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

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where $\Omega_T = \Omega\times]0,T[$, $\Sigma_T = \Gamma\times]0,T[$. Here, $\boldsymbol{u}$ denotes the velocity field and p is the scaler pressure. The constant $\gamma > 0$ is the Brinkman coefficient, $a > 0$ is the Darcy coefficient, $b > 0$ is the Forchheimer coefficient and $\alpha \in [1,2]$ is a given number. Ω is a bounded domain of $\mathbb{R}^d$ $(d = 2,3)$, with a sufficiently smooth boundary Γ. Δ is the Laplace operator.

Several papers are devoted to the mathematical study of (DBF) equations. The results concerning the structural stability for the coefficients in flows can be found in articles of [1–3]. In [4], the asymptotic behavior of solutions is investigated. In [5], a perturbed compressible system that approximate the Brinkman-Forchheimer equations is analyzed. The existence and uniqueness of a weak solution is established and also how the solution of the perturbed problem converges to the solution of the Brinkman-Forchheimer problem. The existence of regular dissipative solutions with the nonlinearity of an arbitrary polynomial growth rate is examined in [6].

The primary difficulty in computing incompressible flows is in finding a satisfactory way to link changes in the velocity field to the pressure variation. Commonly used methods are the pseudocompressibility method. Pseudocompressibility methods relax the incompressibility constraint by perturbing it in an appropriate manner. The first convergence results for these methods are given by Chorin [7, 8] and Temam [9] for the Navier-stokes equations. See also [10–12]. In this work, we consider a particular pseudocompressibility approach to compute (DBF) problems. More precisely, we consider the following approximate method with the parameters $\epsilon > 0$:

$$\begin{gathered} \boldsymbol{u}_t^\epsilon - \gamma\Delta\boldsymbol{u}^\epsilon + a\boldsymbol{u}^\epsilon + b|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon + \nabla p^\epsilon = \boldsymbol{f} \quad \text{in } \Omega_T, \\ \operatorname{div}\boldsymbol{u}^\epsilon - \epsilon\Delta p^\epsilon = 0 \quad \text{in } \Omega_T, \end{gathered} \tag{1.2}$$

associated with the following boundary conditions and initial data

$$\begin{gathered} \boldsymbol{u}^\epsilon = \boldsymbol{0} \quad \text{and} \quad \frac{\partial p^\epsilon}{\partial n} = 0 \quad \text{on } \Sigma_T \\ \boldsymbol{u}^\epsilon(0) = \boldsymbol{u}_0 \quad \text{in } \Omega, \end{gathered} \tag{1.3}$$

The article is organized as follows. In Section II, we introduce some notations and preliminary results. In Section III, error estimates for both linear and nonlinear perturbed problems are presented. The time discrete approximation procedure of the perturbed problem is analyzed in Section IV. Our main result shows that the truncation error associated with the proposed scheme is of second order in time. In Section V, a time dicretization of the perturbed system combined with a finite element space discretization are implemented and the performance of the presented method is illustrated.

II. NOTATIONS AND PRELIMINARIES

In this section, we introduce some notations and preliminary results that will be used in the next sections.

We note $\boldsymbol{H}^s(\Omega)$ the classical Sobolev space, and $\|\cdot\|_s$ the associated norm. The norm of a function in $\boldsymbol{L}^2(\Omega)$ is denoted $\|\cdot\|$. We define the spaces: $\boldsymbol{V} = \left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \operatorname{div}\boldsymbol{v} = 0\right\}$ and

$$\boldsymbol{H} = \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \operatorname{div}\boldsymbol{v} = 0, \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\right\}.$$

We will denote by $(\cdot,\cdot)$ the inner product in $\boldsymbol{H}$. As usual, the dual space of $\boldsymbol{H}_0^1(\Omega)$ will be denoted by $\boldsymbol{H}^{-1}(\Omega)$ and $<\cdot,\cdot>$ the dual pairing between the latter two spaces.

C is the generic constant that can take different values in different places. We note that the mapping $\boldsymbol{F}: \boldsymbol{x} \mapsto |\boldsymbol{x}|^\alpha \boldsymbol{x}$ is monotone, then:

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}, \quad (|\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{v}|^\alpha \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}) \geq 0. \tag{2.1}$$

We denote by $D(A) = \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ the domain of the Stokes operator A defined by

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}, \quad (A\boldsymbol{u}, \boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}).$$

We define the inverse Stokes operator: $A^{-1} : \boldsymbol{H}^{-1}(\Omega) \to \boldsymbol{V}$ as follows: for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $(\boldsymbol{v} = A^{-1}\boldsymbol{u}, \theta) \in \boldsymbol{V} \times L_0^2(\Omega)$ is the solution of the following problem:

$$\begin{cases} (\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) - (\theta, \operatorname{div} \boldsymbol{w}) = <\boldsymbol{u}, \boldsymbol{w}>, & \forall \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \\ (q, \operatorname{div} \boldsymbol{v}) = 0, & \forall q \in L_0^2(\Omega), \end{cases} \tag{2.2}$$

where $<\cdot, \cdot>$ is the duality pairing between $\boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$. We recall the following properties of the operator A^{-1} (see [13]):

For any $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{H}^{-1}(\Omega) \times \boldsymbol{H}^{-1}(\Omega)$, the bilinear form $(\boldsymbol{u}, \boldsymbol{v}) \mapsto < A^{-1}\boldsymbol{u}, \boldsymbol{w} >$ induces a seminorm on $\boldsymbol{H}^{-1}(\Omega)$ and we have the following estimate:

$$C_1\|\boldsymbol{u}\|_{-1} \leq \|A^{-1}\boldsymbol{u}\|_1 \leq C_2\|\boldsymbol{u}\|_{-1}. \tag{2.3}$$

It is well known [5, 6, 14] that for $\boldsymbol{u}_0 \in \boldsymbol{V}$ and $\boldsymbol{f} \in \boldsymbol{L}^2(0, T, \boldsymbol{L}^2(\Omega))$, the weak solution of problem (1.1) is smooth and satisfies the following estimates: for any $T > 0$

$$\sup_{0 \leq t \leq T} \|\nabla \boldsymbol{u}(t)\| \leq C \quad \text{and} \quad \int_0^T \|\boldsymbol{u}_t(t)\|^2 \mathrm{d}t \leq C, \tag{2.4}$$

where C is a positive constant depending on $\boldsymbol{u}_0$ and the parameters of problem (1.1). Moreover, if the solution $(\boldsymbol{u}, p)$ is smooth the following estimate holds (see [6, Theorem 2.7]):

$$\sup_{t \in [0, T]} \{\|\boldsymbol{u}(t)\|_2 + \|\nabla p(t)\|\} \leq C. \tag{2.5}$$

In this section, we review some results which will be used in Sections III and IV.

Lemma 2.1. *Let $\boldsymbol{u}$, $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ satisfying (2.4). Then, $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ satisfies*

$$\|F(\boldsymbol{u}) - F(\boldsymbol{v}))\| \leq C\|\nabla \boldsymbol{w}\| \tag{2.6}$$

Proof. We use the first-order Taylor expansion

$$\boldsymbol{F}(\boldsymbol{u}) - \boldsymbol{F}(\boldsymbol{v}) = \int_0^1 \boldsymbol{DF}(\boldsymbol{u}^\theta).\boldsymbol{w} d\theta, \quad \boldsymbol{u}^\theta = \boldsymbol{v} + \theta(\boldsymbol{u} - \boldsymbol{v}),$$

where the Frechet derivative is given by

$$\boldsymbol{DF}(\boldsymbol{u}).h = (\alpha|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u} + |\boldsymbol{u}|^\alpha|).h,$$

and satisfies

$$|\boldsymbol{DF}(\boldsymbol{u}).h| \leq (\alpha+1)|\boldsymbol{u}|^{\alpha}|h|.$$

In particular, we get

$$\|\boldsymbol{F}(\boldsymbol{u}) - \boldsymbol{F}(\boldsymbol{v})\| \leq (\alpha+1)(\|\, |\boldsymbol{u}|^{\alpha}|\boldsymbol{w}| + |\boldsymbol{v}|^{\alpha}|\boldsymbol{w}|\, \|).$$

Thanks to Hölder inequality

$$\|\, |\boldsymbol{u}|^{\alpha}|\boldsymbol{w}|\, \| \leq \|\boldsymbol{u}\|^{\alpha}_{\alpha p_1}\|\boldsymbol{w}\|_{\frac{2p_1}{p_1-2}}.$$

We choose p_1 such that $\alpha p_1 \leq 6$ and $\frac{2p_1}{p_1-2} \leq 6$.

Collecting the above estimates and using (2.4), we find

$$\begin{aligned}\|\boldsymbol{F}(\boldsymbol{u}) - \boldsymbol{F}(\boldsymbol{v})\| &\leq (\alpha+1)\left(\|\boldsymbol{u}\|_6^{\alpha}\,\|\boldsymbol{w}\|_{\frac{6}{3-\alpha}} + \|\boldsymbol{v}\|_6^{\alpha}\,\|\boldsymbol{w}\|_{\frac{6}{3-\alpha}}\right)\\ &\leq (\alpha+1)\kappa^{\alpha+1}(\|\nabla\boldsymbol{u}\|^{\alpha} + \|\nabla\boldsymbol{v}\|^{\alpha})\|\boldsymbol{w}\|_{\frac{6}{3-\alpha}}\\ &\leq 2(\alpha+1)\kappa^{\alpha+1}C^{\alpha}\|\nabla\boldsymbol{w}\|,\end{aligned}$$

where κ is the constant in the Sobolev inequality:

$$\forall\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \|\boldsymbol{v}\|_p \leq \kappa\|\nabla\boldsymbol{v}\| \quad \text{for any } 2 \leq p \leq 6, \tag{2.7}$$

■

The next Lemma gives the control of $\boldsymbol{u}_{tt}$.

Lemma 2.2. *Let $(\boldsymbol{u}, p)$ be a sufficiently regular solution of problem (1.1). Assume that $\boldsymbol{f}_t \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, then we have:*

$$\int_0^T \|\boldsymbol{u}_{tt}\|_{-1}dt \leq C. \tag{2.8}$$

Proof. Let us differentiate $(1.1)^1$ with respect to time, multiplying by $A^{-1}\boldsymbol{u}_{tt}$, integrating over Ω and using (2.3), we arrive at:

$$\|\boldsymbol{u}_{tt}\|^2_{-1} + \frac{\gamma}{2}(\|\boldsymbol{u}_t\|^2)_t \leq b|(F'(\boldsymbol{u})\boldsymbol{u}_t, A^{-1}\boldsymbol{u}_{tt})| + a|(\boldsymbol{u}_t, A^{-1}\boldsymbol{u}_{tt})| + |(\boldsymbol{f}_t, A^{-1}\boldsymbol{u}_{tt})|. \tag{2.9}$$

Thus, making use of Hölder's inequality and Sobolev inequalities in the first term in the right-hand side of (2.9), one obtains:

$$\begin{aligned}b|(F'(\boldsymbol{u})\boldsymbol{u}_t, A^{-1}\boldsymbol{u}_{tt})| &\leq (\alpha+1)b|(|\boldsymbol{u}|^{\alpha}|\boldsymbol{u}_t|, |A^{-1}\boldsymbol{u}_{tt}|)|\\ &\leq C(\alpha+1)b\|\boldsymbol{u}\|^{\alpha}_{3\alpha}\|\boldsymbol{u}_t\|\|\boldsymbol{u}_{tt}\|_{-1}.\end{aligned}$$

By the use of (2.4) and (2.7), it follows

$$b|(F'(\boldsymbol{u})\boldsymbol{u}_t, A^{-1}\boldsymbol{u}_{tt})| \leq C\|\boldsymbol{u}_t\|^2 + \frac{1}{4}\|\boldsymbol{u}_{tt}\|^2_{-1}, \tag{2.10}$$

where the constant C depends on b, α, κ. Similarly,

$$a|(\boldsymbol{u}_t, A^{-1}\boldsymbol{u}_{tt})| \leq Ca\|\boldsymbol{u}_t\|\|\boldsymbol{u}_{tt}\|_{-1}$$
$$\leq C\|\boldsymbol{u}_t\|^2 + \frac{1}{4}\|\boldsymbol{u}_{tt}\|_{-1}^2,$$

where the constant C depends on a. For the last term in (2.9), we have:

$$|(\boldsymbol{f}_t, A^{-1}\boldsymbol{u}_{tt})| \leq C\|\boldsymbol{f}_t\|_{-1}^2 + \frac{1}{4}\|\boldsymbol{u}_{tt}\|_{-1}^2.$$

Combining the above estimates and (2.9), we obtain:

$$\frac{1}{4}\|\boldsymbol{u}_{tt}\|_{-1}^2 + \frac{\gamma}{2}\frac{\partial}{\partial t}\|\boldsymbol{u}_t\|^2 \leq C\|\boldsymbol{u}_t\|^2 + C\|\boldsymbol{f}_t\|_{-1}^2.$$

Integrating over $[0, T]$ and using (2.4), we arrive at (2.8) and finish the proof of the lemma. ■

III. ERROR ESTIMATES FOR THE PRESSURE STABILIZATION METHOD

The existence of weak solutions of the proposed approximation method (1.2) as well as the convergence of the approximate solution $\boldsymbol{u}^\epsilon$ to the solution $\boldsymbol{u}$ of the initial (DBF) problem can be proved using standard techniques. The reader is referred to, for example, [5, 7]. The main result of this section is stated in Theorem 3.5 where we derive error estimates for the perturbed system (1.2). The proof is split into two steps: In the first one, the error related to the linear case is given. The second step concerns the error behavior for the fully nonlinear problem.

A. The Linearly Perturbed Problem

Let $(\boldsymbol{u}, p)$ the solution of the Darcy-Brinkman-Forchheimer (DBF) equations and we consider the linearly perturbed problem:

$$\begin{aligned} \boldsymbol{v}_t^\epsilon - \gamma\Delta\boldsymbol{v}^\epsilon + a\boldsymbol{v}^\epsilon + \nabla r^\epsilon &= \boldsymbol{f} - b|\boldsymbol{u}|^\alpha\boldsymbol{u} \quad \text{in } \Omega_T, \\ \operatorname{div}\boldsymbol{v}^\epsilon - \epsilon\Delta r^\epsilon &= 0 \quad \text{in } \Omega_T, \end{aligned} \tag{3.1}$$

with the boundary and initial conditions:

$$\begin{aligned} \boldsymbol{v}^\epsilon = \boldsymbol{0} \quad \text{and} \quad \frac{\partial r^\epsilon}{\partial n} = 0 \quad \text{on } \Sigma_T \\ \boldsymbol{v}^\epsilon(0) = \boldsymbol{u}_0 \quad \text{and} \quad r^\epsilon(0) = p(0) \quad \text{in } \Omega, \end{aligned} \tag{3.2}$$

Let $\boldsymbol{\xi} = \boldsymbol{u} - \boldsymbol{v}^\epsilon$ and $\psi = p - r^\epsilon$ and substracting (3.1) from (1.1), we obtain:

$$\begin{aligned} \partial_t\boldsymbol{\xi} - \gamma\Delta\boldsymbol{\xi} + a\boldsymbol{\xi} + \nabla\psi &= \boldsymbol{0} \quad \text{in } \Omega_T, \\ \operatorname{div}\boldsymbol{\xi} - \epsilon\Delta\psi &= -\epsilon\Delta p \quad \text{in } \Omega_T. \end{aligned} \tag{3.3}$$

Moreover, using (1.3) and (3.2), we have:

$$\begin{aligned} \boldsymbol{\xi} = \boldsymbol{0} \quad \text{and} \quad \frac{\partial\psi}{\partial n} = \frac{\partial p}{\partial n} \quad \text{on } \Sigma_T \\ \boldsymbol{\xi}(0) = \boldsymbol{0} \quad \text{and} \quad \psi(0) = 0 \quad \text{in } \Omega. \end{aligned} \tag{3.4}$$

We begin by proving the following estimates

Lemma 3.1. *Assuming (2.5), we have:*

$$\|\boldsymbol{\xi}(t)\|^2 + \inf(\gamma, a)\int_0^t \|\boldsymbol{\xi}(s)\|_1^2 ds + \epsilon \int_0^t \|\nabla\psi(s)\|^2 ds \leq C\epsilon. \tag{3.5}$$

$$\int_0^t \|\boldsymbol{\xi}(s)\|^2 ds \leq C\epsilon^2. \tag{3.6}$$

Proof. Multiplying the Eq. $(3.3)^1$ by $\boldsymbol{\xi}$ and the Eq. $(3.3)^2$ by ψ, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\xi}\|^2 + \gamma\|\nabla\boldsymbol{\xi}\|^2 + a\|\boldsymbol{\xi}\|^2 + \epsilon\|\nabla\psi\|^2 = \epsilon(\nabla p, \nabla\psi)$$
$$\leq \frac{\epsilon}{2}\|\nabla p\|^2 + \frac{\epsilon}{2}\|\nabla\psi\|^2,$$

so,

$$\frac{d}{dt}\|\boldsymbol{\xi}\|^2 + 2\gamma\|\nabla\boldsymbol{\xi}\|^2 + 2a\|\boldsymbol{\xi}\|^2 + \epsilon\|\nabla\psi\|^2 \leq \epsilon\|\nabla p\|^2.$$

Integrating from 0 to t, as $\boldsymbol{\xi}(0) = \mathbf{0}$ and using (2.5), we obtain:

$$\|\boldsymbol{\xi}(t)\|^2 + \inf(\gamma, a)\int_0^t \|\boldsymbol{\xi}(s)\|_1^2 ds + \epsilon\int_0^t \|\nabla\psi(s)\|^2 ds \leq \epsilon\int_0^t \|\nabla p(s)\|^2 ds \leq C\epsilon,$$

which is our claim (3.5).

To deduce the assertion (3.6), we need to introduce the following auxiliary problem: let $(\boldsymbol{w}, q)$ be a strong solution of:

$$\begin{aligned} \boldsymbol{w}_t + \gamma\Delta\boldsymbol{w} - a\boldsymbol{w} - \nabla q = \boldsymbol{\xi}(t) \quad \text{in } \Omega_T, \quad \forall 0 \leq t \leq s, \\ \operatorname{div}\boldsymbol{w} = 0 \quad \text{in } \Omega_T, \end{aligned} \tag{3.7}$$

with the associate boundary and initial conditions:

$$\begin{aligned} \boldsymbol{w} = \mathbf{0} \quad \text{on } \Sigma_T \\ \boldsymbol{w}(s) = \mathbf{0} \quad \text{in } \Omega, \quad (s \leq T). \end{aligned} \tag{3.8}$$

The below inequality can be shown by a standard procedure:

$$\int_0^s (\|\boldsymbol{w}\|_2^2 + \|\nabla q\|^2)\mathrm{d}t \leq C\int_0^s \|\boldsymbol{\xi}(t)\|^2 \mathrm{d}t. \tag{3.9}$$

Taking the inner product of $(3.7)^1$ with $\boldsymbol{\xi}(t)$, as $\boldsymbol{\xi}$ is solution of (3.3) and $\operatorname{div}\boldsymbol{w} = 0$, we obtain:

$$\begin{aligned} \|\boldsymbol{\xi}(t)\|^2 &= \frac{d}{dt}(\boldsymbol{w}, \boldsymbol{\xi}(t)) - (\boldsymbol{w}, \boldsymbol{\xi}_t(t)) - \gamma(\nabla\boldsymbol{w}, \nabla\boldsymbol{\xi}(t)) - a(\boldsymbol{w}, \boldsymbol{\xi}(t) + (\nabla q, \boldsymbol{\xi}(t)) \\ &= \frac{d}{dt}(\boldsymbol{w}, \boldsymbol{\xi}(t)) + (\nabla\psi, \boldsymbol{w}) - (q, \operatorname{div}\boldsymbol{\xi}(t)) \\ &= \frac{d}{dt}(\boldsymbol{w}, \boldsymbol{\xi}(t)) + \epsilon(q, \Delta p) - \epsilon(q, \Delta\psi) \\ &= \frac{d}{dt}(\boldsymbol{w}, \boldsymbol{\xi}(t)) + \epsilon(q, \Delta r^\epsilon) = \frac{d}{dt}(\boldsymbol{w}, \boldsymbol{\xi}(t)) - \epsilon(\nabla q, \nabla r^\epsilon). \end{aligned}$$

Integrating from 0 to s, as $\boldsymbol{w}(s)=\mathbf{0}$ and $\boldsymbol{\xi}(0)=\mathbf{0}$, using (3.9) we have for $\epsilon'>0$:

$$\begin{aligned}\int_0^s \|\boldsymbol{\xi}(t)\|^2 \mathrm{d}t &\le \epsilon \int_0^s \|\nabla q\|\|\nabla r^\epsilon\| \mathrm{d}t \\ &\le \epsilon' \int_0^s \|\nabla q\|^2 \mathrm{d}t + \frac{\epsilon^2}{\epsilon'} \int_0^s \|\nabla r^\epsilon\|^2 \mathrm{d}t, \\ &\le C\epsilon' \int_0^s \|\boldsymbol{\xi}(t)\|^2 \mathrm{d}t + \frac{\epsilon^2}{\epsilon'} \int_0^s \|\nabla r^\epsilon\|^2 \mathrm{d}t.\end{aligned}$$

We conclude by choosing $\epsilon' < \frac{1}{C}$ and using (2.5), (3.5). ■

Lemma 3.2. *In addition to the assumption (2.5), we assume that*

$$\int_0^T t^2 \|\nabla p_t(t)\|^2 dt \le M. \tag{3.10}$$

Then, we have for any $0 \le t \le T$*:*

$$t\|\boldsymbol{\xi}\|_1^2 + t^2\|\psi\|^2 \le C\epsilon \tag{3.11}$$

$$t\|\boldsymbol{\xi}\|^2 \le C\epsilon^2. \tag{3.12}$$

Proof. We differentiate $(3.3)^2$ and $(3.4)^1$ with respect to t, we obtain:

$$\operatorname{div}\boldsymbol{\xi}_t - \epsilon\Delta\psi_t = -\epsilon\Delta p_t, \quad \text{in } \Omega_T, \quad \frac{\partial \psi_t}{\partial n} = \frac{\partial p_t}{\partial n} \quad \text{on } \Sigma_T. \tag{3.13}$$

Taking the scalar product of (3.13) with $2t\psi$ and of $(3.3)^1$ with $2t\boldsymbol{\xi}_t$, summing up, we get:

$$\begin{aligned}&2t\|\boldsymbol{\xi}_t\|^2 + \gamma\frac{d}{dt}(t\|\nabla\boldsymbol{\xi}\|^2) + a\frac{d}{dt}(t\|\boldsymbol{\xi}\|^2) + \epsilon\frac{d}{dt}(t\|\nabla\psi\|^2) \\ &= \gamma\|\nabla\boldsymbol{\xi}\|^2 + a\|\boldsymbol{\xi}\|^2 + \epsilon\|\nabla\psi\|^2 + 2\epsilon t(\nabla p_t, \nabla\psi) \\ &\le \gamma\|\nabla\boldsymbol{\xi}\|^2 + a\|\boldsymbol{\xi}\|^2 + 2\epsilon\|\nabla\psi\|^2 + \epsilon t^2\|\nabla p_t\|^2.\end{aligned} \tag{3.14}$$

Integrating (3.14) from 0 to t, by the help of (3.5) and (3.10), we obtain:

$$2\int_0^t s\|\boldsymbol{\xi}_t\|^2 \mathrm{d}s + \inf(\gamma, a) t\|\boldsymbol{\xi}(t)\|_1^2 + \epsilon t\|\nabla\psi(t)\|^2 \le C\epsilon + \epsilon\int_0^t s^2\|\nabla p_t\|^2 \mathrm{d}s \le C\epsilon.$$

In particular, we have:

$$\int_0^t s\|\boldsymbol{\xi}_t\|^2 \mathrm{d}s + t\|\boldsymbol{\xi}\|_1^2 \le C\epsilon, \tag{3.15}$$

which gives the first part of the assertion (3.11).

Now, we need to verify the following bound:

$$t^2\|\boldsymbol{\xi}_t\|^2 + \epsilon\int_0^t s^2\|\nabla\psi_t\|^2 \mathrm{d}s \le C\epsilon. \tag{3.16}$$

For this purpose, we differentiate (3.3)[1] with respect to t, we obtain:

$$\boldsymbol{\xi}_{tt} - \gamma \Delta \boldsymbol{\xi}_t + a\boldsymbol{\xi}_t + \nabla \psi_t = 0. \tag{3.17}$$

Taking the inner product of (3.17) with $t^2\boldsymbol{\xi}_t(t)$ and (3.13) with $t^2\psi_t(t)$, we obtain by adding them up:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(t^2\|\boldsymbol{\xi}_t\|^2) + \gamma t^2\|\nabla\boldsymbol{\xi}_t\|^2 + at^2\|\boldsymbol{\xi}_t\|^2 + \epsilon t^2\|\nabla\psi_t\|^2 \\
&\quad = t\|\boldsymbol{\xi}_t\|^2 + \epsilon t^2(\nabla p_t, \nabla\psi_t) \le t\|\boldsymbol{\xi}_t\|^2 + \frac{\epsilon t^2}{2}\|\nabla p_t\|^2 + \frac{\epsilon t^2}{2}\|\nabla\psi_t\|^2.
\end{aligned}$$

Integrating the above inequality from 0 to t, using (3.15) and (3.10), we obtain:

$$\begin{aligned}
&t^2\|\boldsymbol{\xi}_t(t)\|^2 + 2\int_0^t (\gamma s^2\|\nabla\boldsymbol{\xi}_t(s)\|^2 + as^2\|\boldsymbol{\xi}_t(s)\|^2)\mathrm{d}s + \epsilon\int_0^t s^2\|\nabla\psi_t(s)\|^2\mathrm{d}s \\
&\quad \le 2\int_0^t s\|\boldsymbol{\xi}_t(s)\|^2\mathrm{d}s + \epsilon\int_0^t s^2\|\nabla p_t(s)\|^2\mathrm{d}s \le C\epsilon,
\end{aligned}$$

which completes the proof of (3.16).

Now, we are in position to show (3.12). We take the scalar product of (3.7) with $t\boldsymbol{\xi}_t$:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(t\|\boldsymbol{\xi}\|^2) &= \frac{1}{2}\|\boldsymbol{\xi}\|^2 + (\boldsymbol{\xi}_t, t\boldsymbol{w}_t) + t\gamma(\Delta\boldsymbol{\xi}_t, \boldsymbol{w}) - at(\boldsymbol{\xi}_t, \boldsymbol{w}) + t(q, \operatorname{div}\boldsymbol{\xi}_t) \\
&= \frac{1}{2}\|\boldsymbol{\xi}\|^2 + (\boldsymbol{\xi}_t, t\boldsymbol{w}_t) + \gamma\frac{d}{dt}(t(\Delta\boldsymbol{\xi}, \boldsymbol{w})) - \gamma(\Delta\boldsymbol{\xi}, t\boldsymbol{w}_t) - \gamma(\boldsymbol{\xi}, \Delta\boldsymbol{w}) \\
&\quad - a\frac{d}{dt}(t(\boldsymbol{\xi}, \boldsymbol{w})) + a(\boldsymbol{\xi}, \boldsymbol{w}) + a(\boldsymbol{\xi}, t\boldsymbol{w}_t) + \epsilon t(\nabla q, \nabla r_t^\epsilon) \\
&= \frac{1}{2}\|\boldsymbol{\xi}\|^2 + \gamma\frac{d}{dt}t(\Delta\boldsymbol{\xi}, \boldsymbol{w}) - \gamma(\boldsymbol{\xi}, \Delta\boldsymbol{w}) - a\frac{d}{dt}t(\boldsymbol{\xi}, \boldsymbol{w}) + a(\boldsymbol{\xi}, \boldsymbol{w}) \\
&\quad + (\nabla\psi, t\boldsymbol{w}_t) + \epsilon t(\nabla q, \nabla r_t^\epsilon)
\end{aligned}$$

So, as $\operatorname{div}\boldsymbol{w}_t = 0$, we can write:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(t\|\boldsymbol{\xi}_t\|^2) &= \frac{1}{2}\|\boldsymbol{\xi}\|^2 + \gamma\frac{d}{dt}t(\Delta\boldsymbol{\xi}, \boldsymbol{w}) - \gamma(\boldsymbol{\xi}, \Delta\boldsymbol{w}) - a\frac{d}{dt}t(\boldsymbol{\xi}, \boldsymbol{w}) + a(\boldsymbol{\xi}, \boldsymbol{w}) + \epsilon t(\nabla q, \nabla r_t^\epsilon) \\
&\le \left(\frac{1}{2} + a + \gamma\right)\|\boldsymbol{\xi}\|^2 + (\gamma + a)\|\boldsymbol{w}\|_2^2 + \gamma\frac{d}{dt}t(\Delta\boldsymbol{\xi}, \boldsymbol{w}) - a\frac{d}{dt}t(\boldsymbol{\xi}, \boldsymbol{w}) \\
&\quad + \frac{1}{2}\|\nabla q\|^2 + \frac{1}{2}\epsilon^2 t^2\|\nabla r_t^\epsilon\|^2.
\end{aligned}$$

Integrating from 0 to s, using (3.6) and (3.9), we arrive at:

$$\begin{aligned}
s\|\boldsymbol{\xi}(s)\|^2 &\le C(a,\gamma)\epsilon^2 + C\epsilon^2\int_0^s t^2\|\nabla r_t^\epsilon\|^2\mathrm{d}t \\
&\le C(a,\gamma)\epsilon^2 + C\epsilon^2\int_0^s t^2\|\nabla p_t\|^2\mathrm{d}t + C\epsilon^2\int_0^s t^2\|\nabla\psi_t\|^2\mathrm{d}t.
\end{aligned}$$

The proof is straightforward by using (3.16) and (3.10).

It remains to prove the pressure estimate. We can use the equation:

$$\nabla\psi = -\boldsymbol{\xi}_t + \gamma\Delta\boldsymbol{\xi} - a\boldsymbol{\xi},$$

and the fact that,

$$\|\psi\| \le C\|\nabla\psi\|_{-1}.$$

Then, using (3.16) and (3.15), we arrive at:

$$t^2\|\psi\|^2 \le Ct^2(\|\boldsymbol{\xi}_t\|^2 + \|\boldsymbol{\xi}\|_1^2) \le C\epsilon$$

■

B. Error Estimate for the Nonlinear Perturbed Problem

Let $(\boldsymbol{v}^\epsilon, r^\epsilon)$ be the solution of the system (3.1) and $(\boldsymbol{u}^\epsilon, p^\epsilon)$ the solution of problem (1.2). Letting $\boldsymbol{\eta} = \boldsymbol{v}^\epsilon - \boldsymbol{u}^\epsilon$ and $\phi = r^\epsilon - p^\epsilon$ and subtracting (1.2) from (3.1), we obtain:

$$\begin{gathered}
\boldsymbol{\eta}_t - \gamma\Delta\boldsymbol{\eta} + a\boldsymbol{\eta} + \nabla\phi = |\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon - |\boldsymbol{u}|^\alpha\boldsymbol{u} \quad \text{in } \Omega_T,\\
\operatorname{div}\boldsymbol{\eta} - \epsilon\Delta\phi = 0 \quad \text{in } \Omega_T,\\
\boldsymbol{\eta} = \mathbf{0} \quad \text{and} \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Sigma_T\\
\boldsymbol{\eta}(0) = \mathbf{0} \quad \text{in } \Omega,\\
\phi(0) = 0 \quad \text{in } \Omega.
\end{gathered} \tag{3.18}$$

With the same arguments used in Lemma 3.1, we have the following result:

Lemma 3.3. *Assume (2.5). We have*

$$\|\boldsymbol{\eta}(t)\|_2^2 + \inf(\gamma, a)\int_0^t \|\boldsymbol{\eta}(s)\|_1^2 ds + \epsilon\int_0^t \|\nabla\phi(s)\|^2 ds \le C\epsilon. \tag{3.19}$$

Proof. The proof can be obtained in a similar way as in Lemma 3.1. Indeed, by taking the inner product of $(3.18)^1$ by η and $(3.18)^2$ by ϕ, and using the following inequality:

$$\begin{aligned}
(F(\boldsymbol{u}^\epsilon) - F(\boldsymbol{u}), \boldsymbol{\eta}) &\le C\|\nabla(\boldsymbol{\xi} + \boldsymbol{\eta})\|\|\boldsymbol{\eta}\|\\
&\le C\|\nabla\boldsymbol{\xi}\|^2 + \frac{a}{2}\|\boldsymbol{\eta}\|^2 + \frac{\gamma}{2}\|\nabla\boldsymbol{\eta}\|^2 + C\|\boldsymbol{\eta}\|^2,
\end{aligned}$$

estimate (3.19) is then a direct consequence of the Gronwall lemma and the estimate (3.5). ■

Lemma 3.4. *Assume (2.5). Then, we have for any* $0 \le t \le T$*:*

$$t\|\boldsymbol{\eta}\|_1^2 + \int_0^t s^2\|\phi\|^2 ds \le C\epsilon \tag{3.20}$$

Proof. We differentiate $(3.18)^2$ and $(3.18)^3$ with respect to t, obtain:

$$\operatorname{div}\boldsymbol{\eta}_t - \epsilon\Delta\phi_t = 0, \quad \text{in } \Omega_T, \qquad \frac{\partial\phi_t}{\partial n} = 0 \quad \text{on } \Sigma_T. \tag{3.21}$$

Taking the scalar product of (3.21) with $2t\phi$ and $(3.18)^1$ with $2t\boldsymbol{\eta}_t$ and summing up the two relations, obtain,

$$2t\|\boldsymbol{\eta}_t\|^2 + \gamma\frac{d}{dt}(t\|\nabla\boldsymbol{\eta}\|^2) + a\frac{d}{dt}(t\|\boldsymbol{\eta}\|^2) + \epsilon\frac{d}{dt}(t\|\nabla\phi\|^2)$$
$$= \gamma\|\nabla\boldsymbol{\eta}\|^2 + a\|\boldsymbol{\eta}\|^2 + \epsilon\|\nabla\phi\|^2 + 2t(F(\boldsymbol{u}^\epsilon) - F(\boldsymbol{u}), \boldsymbol{\eta}_t). \tag{3.22}$$

The nonlinear term in the right-hand side of (3.22) can be treated as follows.

Using the fact that $\boldsymbol{u} - \boldsymbol{u}^\epsilon = \boldsymbol{\xi} + \boldsymbol{\eta}$ and (2.6), we can write:

$$2t(F(\boldsymbol{u}^\epsilon) - F(\boldsymbol{u}), \boldsymbol{\eta}_t) \le Ct\|\nabla(\boldsymbol{\xi} + \boldsymbol{\eta})\|\|\boldsymbol{\eta}_t\|$$
$$\le Ct\|\nabla\boldsymbol{\xi}\|^2 + t\|\boldsymbol{\eta}_t\|^2 + Ct\|\boldsymbol{\eta}\|^2 \tag{3.23}$$

Combining the above inequality into (3.22), obtain

$$t\|\boldsymbol{\eta}_t\|^2 + \gamma\frac{d}{dt}(t\|\nabla\boldsymbol{\eta}\|^2) + a\frac{d}{dt}(t\|\boldsymbol{\eta}\|^2) + \epsilon\frac{d}{dt}(t\|\nabla\phi\|^2)$$
$$= \gamma\|\nabla\boldsymbol{\eta}\|^2 + a\|\boldsymbol{\eta}\|^2 + \epsilon\|\nabla\phi\|^2 + Ct\|\nabla\boldsymbol{\xi}\|^2 + Ct\|\boldsymbol{\eta}\|^2. \tag{3.24}$$

Integrating (3.24) from 0 to t, using (3.11), (3.19) and the Gronwall lemma, we obtain:

$$\int_0^t s\|\boldsymbol{\eta}_t\|^2\mathrm{d}s + \inf(\gamma, a)t(\|\nabla\boldsymbol{\eta}(t)\|^2 + \|\boldsymbol{\eta}(t)\|^2) + \epsilon t\|\nabla\phi(t)\|^2 \le C\epsilon. \tag{3.25}$$

where C is a constant depending on γ and a, which gives the first part of estimate (3.20).

To derive the estimate for the pressure ϕ, we need the following result:

$$\int_0^t s^2\|\boldsymbol{\eta}_t\|^2 ds \le C\epsilon. \tag{3.26}$$

To prove (3.26), we take the inner product of $(3.18)^1$ with $t^2\boldsymbol{\eta}_t$ and (3.21) with $t^2\phi$, we derive by adding them up:

$$t^2\|\boldsymbol{\eta}_t\|^2 + \frac{\gamma}{2}\frac{d}{dt}(t^2\|\nabla\boldsymbol{\eta}\|^2) + \frac{a}{2}\frac{d}{dt}(t^2\|\boldsymbol{\eta}\|^2) + \frac{\epsilon}{2}\frac{d}{dt}(t^2\|\nabla\phi\|^2)$$
$$= \gamma t\|\nabla\boldsymbol{\eta}\|^2 + at\|\boldsymbol{\eta}\|^2 + \epsilon t\|\nabla\phi\|^2 + t^2(F(\boldsymbol{u}^\epsilon) - F(\boldsymbol{u}), \boldsymbol{\eta}_t).$$

The last term in the right-hand side can be controlled as in (3.23) and then the desired result (3.26) follows from estimates (3.11) and (3.25). Finally, using the equation:

$$\nabla\phi = -\boldsymbol{\eta}_t + \gamma\Delta\boldsymbol{\eta} - a\boldsymbol{\eta} + b(F(\boldsymbol{u}^\epsilon) - F(\boldsymbol{u}))$$

and combining estimates (3.11), (3.25), and (3.26), we arrive at:

$$\int_0^t s^2\|\phi\|^2 ds \le C\int_0^t s^2\|\nabla\phi\|_{-1}^2 ds \le C\int_0^t s^2(\|\boldsymbol{\eta}_t\|^2 + \|\nabla\boldsymbol{\eta}\|^2 + \|\boldsymbol{\eta}\|^2 + b\|\nabla\boldsymbol{\xi}\|^2) \le C\epsilon. \quad \blacksquare$$

Finally, combining Lemmas 3.2 and 3.4, we have proved the following theorem.

Theorem 3.5. *We have the following estimate:*

$$t\|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|_1^2 + \int_0^t s^2\|p(t) - p^\epsilon(t)\|^2 ds \le C\epsilon. \tag{3.27}$$

IV. ANALYSIS OF THE TIME DISCRETISATION SCHEME

In this section, we shall analyze the following time discretization scheme: for a given $\boldsymbol{u}^0 = \boldsymbol{u}_0$, solve successively $\tilde{\boldsymbol{u}}^{n+1}$ and $(\boldsymbol{u}^{n+1}, p^{n+1})$ by:

$$\begin{aligned} &\frac{\tilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{k} - \gamma \Delta \tilde{\boldsymbol{u}}^{n+1} + a\tilde{\boldsymbol{u}}^{n+1} + b|\tilde{\boldsymbol{u}}^{n+1}|^\alpha \tilde{\boldsymbol{u}}^{n+1} = \boldsymbol{f}(t_{n+1}) \\ &\tilde{\boldsymbol{u}}^{n+1}|_\Gamma = 0 \end{aligned} \tag{4.1}$$

$$\begin{aligned} \frac{\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}}{k} + \nabla p^{n+1} = 0, \\ \operatorname{div} \boldsymbol{u}^{n+1} = 0, \quad \boldsymbol{u}^{n+1} \cdot \boldsymbol{n}|_\Gamma = 0. \end{aligned} \tag{4.2}$$

The scheme (4.1)–(4.2) is a semidiscretized version of projection method (see Chorin [7, 8] and Temam [9]), where k is the time step, $t_{n+1} = (n+1)k$ and $\boldsymbol{n}$ is the normal vector to the boundary Γ. In the first step, we solve an intermediate $\tilde{\boldsymbol{u}}^{n+1}$ satisfying the boundary condition but does not satisfy the incompressibility condition. Then, in the second step (projection step), we project $\tilde{\boldsymbol{u}}^{n+1}$ on $\boldsymbol{H}$ to get the velocity approximation $\boldsymbol{u}^{n+1}$ which is divergence free.

Observe that $\boldsymbol{u}^n$ can be eliminated in (4.1)–(4.2) and then we can interpret the scheme as a First-Order time discretization to the perturbed problem (1.2; with $\epsilon = k$).

We use the following notations:

$$\boldsymbol{e}^{n+1} = \boldsymbol{u}(t_{n+1}) - \boldsymbol{u}^{n+1} \quad \text{and} \quad \tilde{\boldsymbol{e}}^{n+1} = \boldsymbol{u}(t_{n+1}) - \tilde{\boldsymbol{u}}^{n+1}.$$

Theorem 4.1. *Under the same hypothesis of Lemma 2.2, we have the following estimates: For any* $0 \le N \le T/k - 1$*:*

$$\begin{aligned} &\|e^{N+1}\|^2 + \|\tilde{e}^{N+1}\|^2 + \gamma k \sum_{n=0}^{N} \left\{ \|\nabla \tilde{e}^{n+1}\|^2 + \|\nabla e^{n+1}\|^2 \right\} \\ &\quad + \sum_{n=0}^{N} \left\{ \|\tilde{e}^{n+1} - e^{n+1}\|^2 + \|\tilde{e}^{n+1} - e^n\|^2 \right\} \le Ck^2, \end{aligned} \tag{4.3}$$

$$ak \sum_{n=0}^{N} \left\{ \|\tilde{e}^{n+1}\|^2 + \|e^{n+1}\|^2 \right\} \le Ck^2, \tag{4.4}$$

$$k \sum_{n=0}^{N} \|p(t_{n+1}) - p^{n+1}\|^2_{L^2(\Omega)/R} \le Ck^2. \tag{4.5}$$

where C *is a constant which depends on* γ*,* b*, and* a*.*

Proof. Substracting $(1.1)^1$ from (4.1) to $(1.1)^2$ from (4.2), we obtain the error equations:

$$\begin{aligned} &\frac{\tilde{\boldsymbol{e}}^{n+1} - \boldsymbol{e}^n}{k} - \gamma \Delta \tilde{\boldsymbol{e}}^{n+1} + a\tilde{\boldsymbol{e}}^{n+1} + b|\boldsymbol{u}(t_{n+1})|^\alpha \boldsymbol{u}(t_{n+1}) - b|\tilde{\boldsymbol{u}}^{n+1}|^\alpha \tilde{\boldsymbol{u}}^{n+1} \\ &= -\nabla p(t_{n+1}) + \boldsymbol{R}_n \end{aligned} \tag{4.6}$$

and

$$\frac{\boldsymbol{e}^{n+1} - \tilde{\boldsymbol{e}}^{n+1}}{k} - \nabla p^{n+1} = 0, \tag{4.7}$$

where $\boldsymbol{R}_n$ is the truncation error defined by:

$$\begin{aligned}&\frac{\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)}{k} - \nu\Delta\boldsymbol{u}(t_{n+1}) + a\boldsymbol{u}(t_{n+1}) + b|\boldsymbol{u}(t_{n+1})|^{\alpha}\boldsymbol{u}(t_{n+1})\\ &\quad = -\nabla p(t_{n+1}) + \boldsymbol{R}_n\end{aligned} \tag{4.8}$$

with

$$\boldsymbol{R}_n = \frac{1}{k}\int_{t_n}^{t_{n+1}} (t - t_n)\boldsymbol{u}_{tt}(t)dt, \tag{4.9}$$

satisfying

$$\|\boldsymbol{R}_n\|_{-1}^2 \leq k^{-2}\|\int_{t_n}^{t_{n+1}} (t - t_n)\boldsymbol{u}_{tt}dt\|_{-1}^2 \leq k\int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_{tt}\|_{-1}^2 dt. \tag{4.10}$$

Taking the inner product of (4.6) with $2k\tilde{\boldsymbol{e}}^{n+1}$ and of (4.7) with $2k\boldsymbol{e}^{n+1}$, using the polarization identity and the monotony given in (2.1), we obtain:

$$\begin{aligned}&(2ka + 1)\|\tilde{\boldsymbol{e}}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2 + \|\tilde{\boldsymbol{e}}^{n+1} - \boldsymbol{e}^n\|^2 + 2k\gamma\|\nabla\tilde{\boldsymbol{e}}^{n+1}\|^2\\ &\quad \leq 2k(\nabla p(t_{n+1}), \tilde{\boldsymbol{e}}^{n+1}) + 2k(\boldsymbol{R}_n, \tilde{\boldsymbol{e}}^{n+1}),\end{aligned} \tag{4.11}$$

and

$$\|\boldsymbol{e}^{n+1}\|^2 - \|\tilde{\boldsymbol{e}}^{n+1}\|^2 + \|\tilde{\boldsymbol{e}}^{n+1} - \boldsymbol{e}^{n+1}\|^2 = 0. \tag{4.12}$$

The terms on the right-hand side of (4.11) can be controlled as follows:

As $\boldsymbol{e}^n$ is divergence free, we have:

$$2k(\nabla p(t_{n+1}), \tilde{\boldsymbol{e}}^{n+1}) \leq 2k^2\|\nabla p(t_{n+1})\|^2 + \frac{1}{2}\|\tilde{\boldsymbol{e}}^{n+1} - \boldsymbol{e}^n\|^2.$$

Using (4.10), we derive

$$2k(\boldsymbol{R}_n, \tilde{\boldsymbol{e}}^{n+1}) \leq \gamma k\|\nabla\tilde{\boldsymbol{e}}^{n+1}\|^2 + Ck^2\int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_{tt}(t)\|_{-1}^2 dt.$$

We can now combine the above estimates to verify the following bound:

$$\begin{aligned}&(2ka + 1)\|\tilde{\boldsymbol{e}}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2 + \frac{1}{2}\|\tilde{\boldsymbol{e}}^{n+1} - \boldsymbol{e}^n\|^2 + k\gamma\|\nabla\tilde{\boldsymbol{e}}^{n+1}\|^2\\ &\quad \leq Ck^2\int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_{tt}(t)\|_{-1}^2 dt + 2k^2\|\nabla p(t_{n+1})\|^2.\end{aligned} \tag{4.13}$$

Taking the sum of (4.12) and (4.13) for $n = 0$ to N with $0 \leq N \leq T/k - 1$, thanks to (2.5) and Lemma 2.2) we arrive at:

$$\|\boldsymbol{e}^{N+1}\|^2 + \sum_{n=0}^{N} \left\{ \|\tilde{\boldsymbol{e}}^{n+1} - \boldsymbol{e}^{n+1}\|^2 + \frac{1}{2}\|\tilde{\boldsymbol{e}}^{n+1} - \boldsymbol{e}^n\|^2 + k\gamma\|\nabla\tilde{\boldsymbol{e}}^{n+1}\|^2 + 2ka\|\tilde{\boldsymbol{e}}^{n+1}\|^2 \right\}$$
$$\leq Ck^2 \left(\int_0^T \|\boldsymbol{u}_{tt}(t)\|_{-1}^2 dt + \sup_{t\in[0,T]} \|\nabla p(t)\|^2 \right) \leq Ck^2. \tag{4.14}$$

According to the projection property, we derive from the above inequality that:

$$k\gamma \sum_{n=0}^{N} \|\nabla \boldsymbol{e}^{n+1}\|^2 \leq Ck\gamma \sum_{n=0}^{N} \|\nabla\tilde{\boldsymbol{e}}^{n+1}\|^2 \leq Ck^2. \tag{4.15}$$

Therefore, we derive from (4.12) and (4.14) that

$$\|\tilde{\boldsymbol{e}}^{N+1}\|^2 = \|\boldsymbol{e}^{N+1}\|^2 + \|\tilde{\boldsymbol{e}}^{N+1} - \boldsymbol{e}^{N+1}\|^2 \leq Ck^2 + \sum_{n=0}^{N} \|\tilde{\boldsymbol{e}}^{n+1} - \boldsymbol{e}^{n+1}\|^2 \leq Ck^2.$$

This completes the proof of (4.3). Next, the estimate (4.4) follows immediately from (4.14) and the projection property.

Now, we want to prove the estimate (4.5) for the pressure. Let us denote $\phi^{n+1} = p(t_{n+1}) - p^{n+1}$. By summing (4.6) and (4.7) and subtracting the result from (4.8), we obtain:

$$\frac{\boldsymbol{e}^{n+1} - \boldsymbol{e}^n}{k} - \gamma\Delta\tilde{\boldsymbol{e}}^{n+1} + a\tilde{\boldsymbol{e}}^{n+1} + b|\boldsymbol{u}(t_{n+1})|^\alpha \boldsymbol{u}(t_{n+1}) - b|\tilde{\boldsymbol{u}}^{n+1}|^\alpha \tilde{\boldsymbol{u}}^{n+1} + \nabla\phi^{n+1} = \boldsymbol{R}_n. \tag{4.16}$$

Multiplication of (4.16) with $\boldsymbol{v}$ in $\boldsymbol{H}_0^1(\Omega)$ gives:

$$(\nabla\phi^{n+1}, \boldsymbol{v}) = (\boldsymbol{R}^n - \frac{1}{k}(\boldsymbol{e}^{n+1} - \boldsymbol{e}^n) + \gamma\Delta\tilde{\boldsymbol{e}}^{n+1} - a\tilde{\boldsymbol{e}}^{n+1} - b(F(\boldsymbol{u}(t_{n+1})) - F(\tilde{\boldsymbol{u}}^{n+1}), \boldsymbol{v})$$
$$\leq (\|\boldsymbol{R}^n\|_{-1} + \frac{1}{k}\|\boldsymbol{e}^{n+1} - \boldsymbol{e}^n\|_{-1} + (\gamma + Cb)\|\nabla\tilde{\boldsymbol{e}}^{n+1}\| + a\|\tilde{\boldsymbol{e}}^{n+1}\|)\|\boldsymbol{v}\|_1.$$

As

$$\|\phi^{n+1}\|_{L^2(\Omega)/R} \leq \sup_{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)} \frac{(\nabla\phi^{n+1}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)}},$$

taking the sum from $n = 0$ to N and using the estimates (4.3), (4.4), and (4.10), we conclude that:

$$k \sum_{n=0}^{T/k-1} \|\phi^{n+1}\|_{L^2(\Omega)/R}^2 \leq Ck \sum_{n=0}^{T/k-1} (\|\boldsymbol{R}^n\|_{-1}^2 + a\|\tilde{\boldsymbol{e}}^{n+1}\|^2 + (\gamma + Cb)\|\nabla\tilde{\boldsymbol{e}}^{n+1}\|^2)$$
$$+ \sum_{n=0}^{T/k-1} \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^n\|_{-1}^2 \leq Ck^2. \quad \blacksquare$$

V. NUMERICAL EXPERIMENTS

In this section, we address the finite element approximation and we carry out numerical experiments for the perturbed Brinkman-Forchheimer equations (1.2) but with nonhomogeneous boundary conditions:

$$\begin{aligned} \boldsymbol{u}_t^\epsilon + \gamma \Delta \boldsymbol{u}^\epsilon + a\boldsymbol{u}^\epsilon + b|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon + \nabla p^\epsilon = \boldsymbol{f}, \quad \text{in } \Omega_T, \\ \nabla \cdot \boldsymbol{u}^\epsilon - \epsilon \Delta p^\epsilon = 0, \quad \text{in } \Omega_T, \end{aligned} \tag{5.1}$$

We now carry out spatial discretization. We consider a regular triangulation $\mathcal{T}_h$ of the domain Ω, depending on a positive parameter $h > 0$, made up of triangles $\mathcal{T}_h$. Let V_h and Q_h represent the finite element spaces which approximate the velocity and pressure fields, respectively.

Let V_h consist of $\mathcal{C}^0$ piecewise polynomial functions $\mathcal{P}^m$, ($m = 2$ in the present simulations) over the triangulation $\mathcal{T}_h$ and we define $\mathbf{V_h} = (V_h)^2$ such that $V_h \subset \boldsymbol{H}_0^1(\Omega)$ and for some $m \geq 2$,

$$\inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \{\|\boldsymbol{v} - \boldsymbol{v}_h\| + h\|\nabla(\boldsymbol{v} - \boldsymbol{v}_h)\|\} \leq Ch^m \|\boldsymbol{v}\|_m, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^m(\Omega), \tag{5.2}$$

Let Q_h consist of $\mathcal{C}^0$ piecewise polynomial functions $\mathcal{P}^k$, ($k = 1$ in the present simulations) over the triangulation $\mathcal{T}_h$ such that $Q_h \subset H^1(\Omega) \cap L_0^2(\Omega)$ and for some $k \geq 1$,

$$\inf_{q_h \in Q_h} \{\|q - q_h\| + h\|\nabla(q - q_h)\|\} \leq Ch^k \|q\|_k, \quad \forall q \in L_0^2(\Omega) \cap H^k(\Omega), \tag{5.3}$$

The Galerkin approximation of the perturbed Brinkman-Forchheimer equations reads (5.1) into a variational formulation:

Find $(\boldsymbol{u}^{\epsilon,h}, p^{\epsilon,h}) \in (\mathbf{V_h},\ Q_h)$ such that for all $\boldsymbol{v}_h \in \mathbf{V_h}$:

$$\begin{aligned} &\frac{d}{dt}\langle \boldsymbol{u}^{\epsilon,h}(t), \boldsymbol{v}_h\rangle + \gamma\langle \nabla \boldsymbol{u}^{\epsilon,h}(t), \nabla \boldsymbol{v}_h\rangle + a\langle \boldsymbol{u}^{\epsilon,h}(t), \boldsymbol{v}_h\rangle \\ &\quad + b\langle |\boldsymbol{u}^{\epsilon,h}(t)|^\alpha \boldsymbol{u}^{\epsilon,h}(t), \boldsymbol{v}_h\rangle - \langle p^{\epsilon,h}(t), \nabla \cdot \boldsymbol{v}_h\rangle = \langle \boldsymbol{f}(t), \boldsymbol{v}_h\rangle, \\ &\langle \nabla \cdot \boldsymbol{u}^{\epsilon,h}(t), q_h\rangle + \epsilon\langle \nabla p^{\epsilon,h}(t), \nabla q_h\rangle = 0, \quad \forall q_h \in Q_h. \end{aligned}$$

For the convergence analysis, we can use similar arguments developed in the paper [15] where the authors show stability, and an optimal error estimates for low order mixed finite element spaces.

We implement the above finite element scheme in FreeFem++. In FreeFem++ a lot of adaptation tools are implemented and based on the Delaunay-Voronoi algorithm with some Metric $\mathcal{M}$. The Hessian error indicator gives the metric in a natural way. The goal of the mesh adaptation is to compute better solutions at low cost. In each time step, a linear algebraic system is solved. The problem for which we present results involves the lid-driven cavity flow (a widely used benchmark case for testing Navier-Stokes flow).

We run a large number of time steps to ensure that we reach the steady state solution.

Figures 1–4 are numerical results of the lid-driven cavity flow applied to our Darcy-Brinkman-Forchheimer system. In all example, we set $\epsilon = 0.000001$, $a = 1$, $\alpha = 0.1$, $\gamma = 1$, and $b = 10$.

Figure 1 shows the x-direction velocity component along the vertical centerline. The present results were compared with Ghia's data (the Reynolds number Re is 100) in [16]. It is found that the result match Ghia's data and verify the correctness of the code.

The use of adaptation techniques allows us to provide fine-scale resolution locally and concentrate numerical effort near important flow features Figs. 3 and 4.

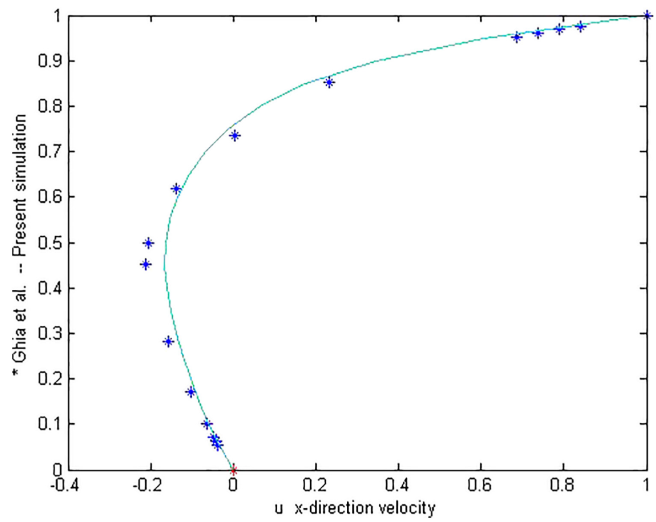

FIG. 1. u component along the vertical line through the cavity center. [Color figure can be viewed at wileyonlinelibrary.com]

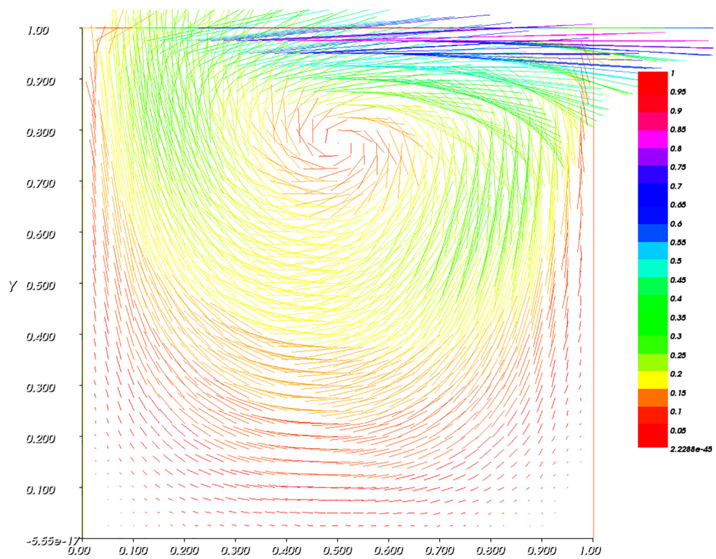

FIG. 2. Velocity fields for the steady solution without mesh adaptation. [Color figure can be viewed at wileyonlinelibrary.com]

VI. CONCLUSION

We have studied a pressure stabilization method scheme in the semidiscretized form for the Brinkman-Forchheimer equations. Error estimates for the velocity and the pressure are established via the energy method. The numerical results presented here indicate that this method is efficient and applicable to classical flow problems.

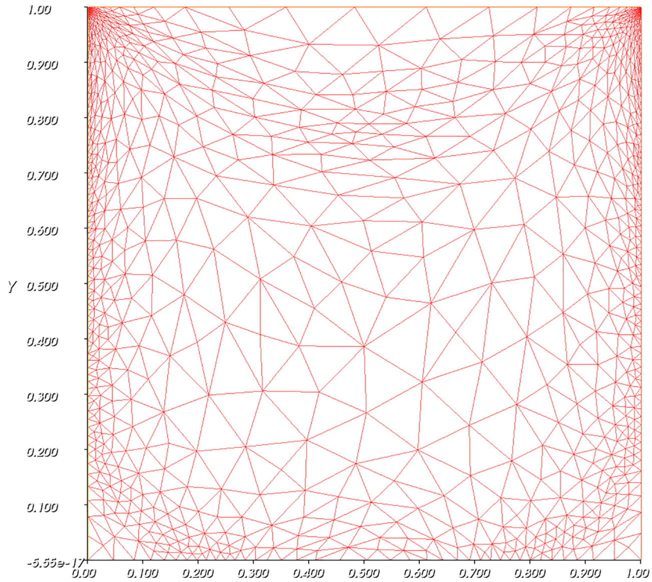

FIG. 3. Final mesh adaption. [Color figure can be viewed at wileyonlinelibrary.com]

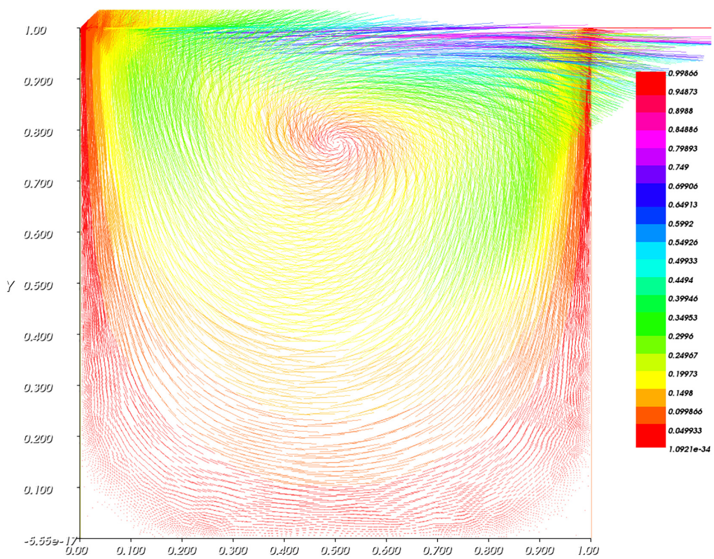

FIG. 4. Associated solution. [Color figure can be viewed at wileyonlinelibrary.com]

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